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A first approximation of concatenated convolutional codes from linear systems theory viewpoint ☆

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Abstract

This article focuses on the characterization of two models of concatenated convolutional codes from the perspective of linear systems theory. We present an input-state-output representation of these models and study the conditions for obtaining a minimal input-state-output representation and non-catastrophic concatenated convolutional code. We also establish conditions so that the concatenated codes are observable and give a lower bound for their free distances.

Keywords: Concatenated convolutional code; Input-state-output representation; Minimal representation; Transfer function; Observable convolutional code; Controllable convolutional code; Observable pair; Controllable pair; Free distance

1. Introduction

Coding theory has arisen from the need for better communication and better computer data storage. Convolutional codes, a class of error correcting codes, are used in many wireless transmissions systems such as transmitting information in deep space with remarkable clarity.

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A mathematical theory has been developed which has a strong relationship with algebra, combinatorics and algebraic geometry. A key problem in convolutional coding theory was to find a method for constructing codes of a given rate and complexity with good free distance. Several methods have been introduced for this task. Perhaps the most popular technique is to relate generator matrices of a convolutional code to generator matrices of some corresponding cyclic or quasi-cyclic code (see for example [14,18,19,29]). Abdel-Ghaffar [1] and Justesen [15] limit their study to convolutional codes of rate $1/n$ and develop very effective techniques for code constructions in this setting. Following this technique, Smarandache et al. [27] give a construction of maximum distance separable convolutional codes for each rate k/n and each degree δ.

It is common knowledge that there is a close connection between linear systems over finite fields and convolutional codes. Rosenthal [22] provides a survey of the different points of view about convolutional codes. Rosenthal, along with York and Schumacher [25,26], introduces the input-state-output representation and gives a construction of a convolutional code with free distance lower-bounded by the complexity of the code, using this representation. Smarandache and Rosenthal [28], make a small adaptation to the construction presented in [25] and introduce a construction of MDS convolutional codes of rate $1/n$.

Another procedure for constructing new high rate convolutional codes from old ones is via puncturing (see for example [21]). This method had widespread success because the best codes constructed by puncturing are typically as powerful as other codes with the same parameters, but are considerably easier to implement than "nonpunctured" codes.

In order to find a class of codes whose probability of error decreased exponentially with code length, while decoding complexity increased only polynomially, Forney [3] came to a solution consisting of the multilevel coding structure known as concatenated code. It consists of a cascade of an inner and an outer code. Berrou et al. [2] introduce an interleaver between the two codes of the concatenation, which provides the correction of error burst from the inner code by the outer code. The result was called "Turbo Codes".

Höst et al. [7,9,10] have developed a new construction of convolutional codes based on code concatenation. This construction consists in a serial concatenation or cascade of convolutional codes, but instead of letting one inner code follow one single outer code, they put a set of parallel codes in place of the outer or the inner code, or both. Since this construction resembles the structure of a fabric, they call these codes *woven* convolutional codes.

Concatenated convolutional codes have always been studied from the generator matrix. In this paper, we introduce a characterization of two kinds of concatenated convolutional codes using linear systems theory. Höst et al. [8] show that the cascade of two canonical generator matrices is not necessarily canonical, and therefore the degree of the convolutional code obtained from the cascade is unknown. In this paper, we achieve conditions for the minimality of an input-state-output representation of the concatenated code, so the degree of the cascade convolutional code can be obtained. Furthermore, these authors give properties of the generator matrix and not for the codes, as we give in this paper. Freudenberger et al. [5], obtain a concatenated code in a binary field with a free distance greater or equal to the product of the free distances of the component codes under additional conditions in terms of the interleavers between the codes. In contrast, working over an arbitrary field (not necessarily binary), we do not only get a lower bound on the free distance of the models of concatenation in terms of the free distance of the outer and inner codes, but also we obtain the degree of the concatenated code. Therefore, we can compare this lower bound with the upper bound given by the generalized Singleton bound.

This paper is structured as follows. In Section 2 we review the way convolutional codes have often been defined in the coding literature. We also explain some recent advances in systems theory in the context of convolutional codes defined over a Galois field. In Section 3 we study a first model of concatenated codes from the point of view of linear systems. We also provide conditions in order to get a minimal input-state-output representation. We give a lower bound of the free distance for these codes. A second model of concatenated codes is developed in Section 4. These models of concatenation do not correspond to the classical series connection in control theory (see for example [16]).

2. Preliminaries

Let $\mathbb{F} = GF(q)$ be the Galois field of q elements, $\mathbb{F}[z]$ the polynomial ring in the variable z with coefficients in $\mathbb{F}$, $\mathbb{F}(z)$ the field of rational functions over $\mathbb{F}$, $\mathbb{F}((z))$ the field of Laurent series and $\overline{\mathbb{F}}$ the algebraic closure of $\mathbb{F}$.

Consider the matrices $A \in \mathbb{F}^{\delta\times\delta}$, $B \in \mathbb{F}^{\delta\times k}$, $C \in \mathbb{F}^{(n-k)\times\delta}$ and $D \in \mathbb{F}^{(n-k)\times k}$. Following [22] and [25], a rate k/n convolutional code $\mathscr{C}$ of degree of complexity δ can be described by the linear system governed by the equations

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t,\\ y_t &= Cx_t + Du_t,\\ v_t &= \begin{pmatrix} y_t \\ u_t \end{pmatrix}, \quad x_0 = 0, \end{aligned} \tag{1}$$

where for each time instant t, $x_t \in \mathbb{F}^{\delta}$ is the *state vector*, $u_t \in \mathbb{F}^k$ is the *information vector*, and $y_t \in \mathbb{F}^{n-k}$ is the *parity vector*. In linear systems theory, this representation is known as the *input-state-output representation*. The integer δ describes the McMillan degree of the linear system (1).

Remark 1. The input-state-output representation (1) is different from a realization often found in the coding literature, where convolutional codes are usually represented by a *driving variable representation* (see [21]),

$$\begin{aligned} x_{t+1} &= \mathscr{A}x_t + \mathscr{B}m_t,\\ v_t &= \mathscr{C}x_t + \mathscr{D}m_t. \end{aligned} \tag{2}$$

Here $m_t \in \mathbb{F}^k$ is the message vector and $v_t \in \mathbb{F}^n$, $x_t \in \mathbb{F}^{\delta}$ as above. This representation was introduced by Massey and Sain [20] and became the standard way in which convolutional codes were represented in linear systems. Rosenthal and York [26] give us good reasons for considering an input-state-output representation for the purpose of constructing convolutional codes.

In terms of an input-state-output representation (1), the free distance of a convolutional code $\mathscr{C}$ can be characterized (see [11]) as

$$d_{\text{free}}(\mathscr{C}) = \min\left(\sum_{t=0}^{\infty} \mathrm{wt}(u_t) + \sum_{t=0}^{\infty} \mathrm{wt}(y_t)\right), \tag{3}$$

where the minimum has to be taken over all possible nonzero codewords and where wt denotes the Hamming weight.

For algebraic reasons we assume that $\{v_t \in \mathbb{F}^n \mid t = 0, 1, 2, \ldots\}$ in Eq. (1) is a finite-weight codeword (see [26]), i.e., Eq. (1) is satisfied for all $t = 0, 1, 2, \ldots$ and there is an integer γ such that $x_{\gamma+1} = 0$, $u_t = 0$, for $t \geqslant \gamma + 1$, and therefore, $y_t = 0$ for $t \geqslant \gamma + 1$, and the code sequence has finite weight. Then, for a finite-weight codeword both the input sequence and the state sequence (and hence the output sequence) need to have finite support. The set of finite-weight codewords has a module structure over the polynomial ring $\mathbb{F}[z]$ (see [26]). By abuse of notation, we will denote this module by $\mathscr{C}(A, B, C, D)$ and we refer it as the *finite-weight convolutional code* generated by the matrices A, B, C, D.

Since $\mathbb{F}[z]$ is a principal ideal domain, $\mathscr{C}(A, B, C, D)$ is a free module of rank k and there is an $n \times k$ polynomial matrix $G(z)$ such that

$$\mathscr{C}(A, B, C, D) = \{v(z) \in \mathbb{F}^n[z] | v(z) = G(z)m(z) \text{ for some } m(z) \in \mathbb{F}^k[z]\}.$$

We call $G(z)$ a *polynomial generator matrix* of the finite-weight convolutional code $\mathscr{C}(A, B, C, D)$.

In the sequel, we will study the properties of finite-weight convolutional codes of the form $\mathscr{C}(A, B, C, D)$. It is customary to define a convolutional code as a $\mathscr{F}$-linear subspace of $\mathscr{F}^n$, where $\mathscr{F}$ is either the field of rational functions $\mathbb{F}(z)$ or the field of Laurent series $\mathbb{F}((z))$ [4,12,13,21]. If $G(z)$ is a polynomial generator matrix of $\mathscr{C}(A, B, C, D)$, then $G(z)$ induces a convolutional code $\widehat{\mathscr{C}} = \widehat{\mathscr{C}}(A, B, C, D) \subset \mathscr{F}^n$ by defining $\widehat{\mathscr{C}}$ as the $\mathscr{F}$-linear span of the columns of $G(z)$ (see [26]). Observe that this definition is independent of the particular generator matrix $G(z)$ of $\mathscr{C}(A, B, C, D)$. The free distance of the convolutional code $\widehat{\mathscr{C}}$ is then defined by expression (3), where the minimization is taken over all possible nonzero codewords in $\widehat{\mathscr{C}}$. By Lemma 2.13 of [26], if the pair (A, C) is observable, the free distance is attained in a finite-weight codeword, that is, in a codeword of $\mathscr{C}(A, B, C, D)$. In fact, McElliece [21] shows that finite-weight codewords are the only ones that can occur in engineering practice. So, in this paper, we will consider

$$d_{\text{free}}(\mathscr{C}) = \lim_{j\to\infty} d_j^c(\mathscr{C}), \tag{4}$$

where

$$d_j^c(\mathscr{C}) = \min_{u_0 \neq 0} \left\{ \sum_{t=0}^{j} \text{wt}(u_t) + \sum_{t=0}^{j} \text{wt}(y_t) \right\} \tag{5}$$

is the jth *column distance* of the convolutional code $\mathscr{C}$ for $j = 0, 1, 2, \ldots$

In the following, we adopt the notation used by McElliece [21] and we call a convolutional code of rate k/n and degree δ an (n, k, δ)-code.

The free distance of an (n, k, δ)-code $\mathscr{C}$ is always upper-bounded (see [24]) by the generalized Singleton bound

$$d_{\text{free}}(\mathscr{C}) \leqslant (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + 1.$$

In addition, the convolutional code $\mathscr{C}$ is called maximum-distance separable (MDS) if its free distance is equal to the generalized Singleton bound.

We define a convolutional code to be *observable* if one, and therefore any, generator matrix $G(z)$ is right prime (see [23]). Furthermore, if $G(z)$ is a generator matrix of an observable convolutional code, then $G(z)$ is a non-catastrophic generator matrix (see [23]).

Rosenthal and York [26] showed that expression (1) describes the state-space realization of a rational and systematic convolutional encoder.

Theorem 1 (Lemma 2.14 of [26]). *Let $\mathscr{C}(A, B, C, D)$ be a convolutional code. Then there are polynomial matrices $Y(z)$, and $U(z)$ of sizes $(n-k) \times k$ and $k \times k$, respectively, such that the matrices A, B, C, and D appearing in* (1) *form a state-space realization of the transfer function $Y(z)U(z)^{-1}$, i.e., one has the relation*

$$C(zI - A)^{-1}B + D = Y(z)U(z)^{-1}.$$

In particular, $T(z) = Y(z)U(z)^{-1}$ *describes a proper transfer function*.

Furthermore, as a generator matrix over $\mathbb{F}^n[z]$, $G(z)$ is equivalent to the systematic generator matrix

$$G_{\text{sys}}(z) = \begin{pmatrix} Y(z)U(z)^{-1} \\ I_k \end{pmatrix}.$$

Note that the description given by system (1) is generally not unique. Sometimes it is possible to describe the code $\mathscr{C}(A, B, C, D)$ using matrices A_1, B_1, C_1, D_1 which are smaller in size than the matrices A, B, C, D. But if $\mathscr{C}$ has degree δ, then it is possible (see [17]) to choose matrices A, B, C, and D of sizes $\delta \times \delta$, $\delta \times k$, $(n-k) \times \delta$ and $(n-k) \times k$, respectively. In convolutional coding theory, an input-state-output representation (A, B, C, D) having the above sizes is called a *minimal representation* and it is characterized through the condition that the pair (A, B) is *controllable*, that is (see [26]),

$$\operatorname{rank}(B \quad AB \quad \cdots \quad A^{\delta-1}B) = \delta$$

or equivalently (see [6]),

$$\operatorname{rank}(zI - A \quad B) = \delta, \quad \text{for all } z \in \overline{\mathbb{F}}.$$

On the other hand, we say that (A, C) is an observable pair if $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is a controllable pair, that is (see [26]),

$$\operatorname{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{\delta-1} \end{pmatrix} = \delta$$

which is equivalent to (see [6]),

$$\operatorname{rank}\begin{pmatrix} zI - A \\ C \end{pmatrix} = \delta, \quad \text{for all } z \in \overline{\mathbb{F}}.$$

Notice that the concept of minimality of an input-state-output representation is different from the concept of minimality of a representation in classical linear systems theory. A representation (A, B, C, D) in linear systems literature is minimal if and only if (A, B) is controllable and (A, C) is observable. In fact, if (A, B) is controllable, then the observability of (A, C) ensures that the linear system (1) describes a noncatastrophic convolutional encoder, as we can see in the following result.

Lemma 1 (Lemma 2.11 of [26]). *Assume that the matrices* (A, B) *form a controllable pair. The convolutional code* $\mathscr{C}(A, B, C, D)$ *defined through* (1) *represents an observable convolutional code if and only if* (A, C) *forms an observable pair*.

3. The first model of concatenated convolutional code

In this section, we introduce our first model of concatenated codes. Let $\mathscr{C}_{\mathrm{o}}$ and $\mathscr{C}_{\mathrm{i}}$ be two convolutional codes that we call *outer code* and *inner code* respectively. In this model, $\mathscr{C}_{\mathrm{o}}$ is

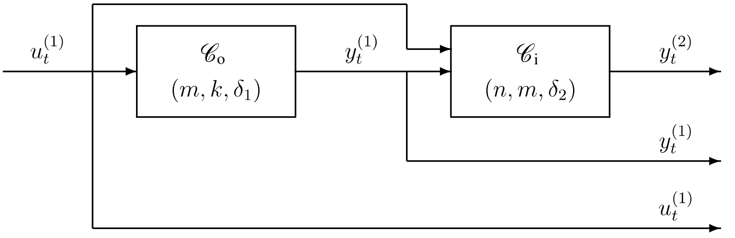

Fig. 1. Concatenated code $\mathscr{C}\mathscr{C}^{(1)}$.

an (m, k, δ_1)-code and $\mathscr{C}_{\mathrm{i}}$ is an (n, m, δ_2)-code. Let $x_t^{(1)}$, $u_t^{(1)}$ and $y_t^{(1)}$ be the state vector, the information vector, and the parity vector of $\mathscr{C}_{\mathrm{o}}$, respectively, and let x_t^2, $u_t^{(2)}$, and $y_t^{(2)}$ be the state vector, the information vector and the parity vector of $\mathscr{C}_{\mathrm{i}}$, respectively. Here the codewords $v_t^{(1)}$ and $v_t^{(2)}$ of $\mathscr{C}_{\mathrm{o}}$ and $\mathscr{C}_{\mathrm{i}}$ respectively, are given by

$$v_t^{(1)} = \begin{pmatrix} y_t^{(1)} \\ u_t^{(1)} \end{pmatrix} \quad \text{and} \quad v_t^{(2)} = \begin{pmatrix} y_t^{(2)} \\ u_t^{(2)} \end{pmatrix}. \tag{6}$$

In this model, the outer code $\mathscr{C}_{\mathrm{o}}$ and the inner code $\mathscr{C}_{\mathrm{i}}$ are serialized, one after the other (see Fig. 1), so that the input information $u_t^{(1)}$ is fed to $\mathscr{C}_{\mathrm{o}}$ and the obtained codeword $v_t^{(1)}$ is then encoded by $\mathscr{C}_{\mathrm{i}}$ in a way that

$$u_t^{(2)} = v_t^{(1)}. \tag{7}$$

We denote by $\mathscr{C}\mathscr{C}^{(1)}$ the corresponding concatenated convolutional code. Note that the vector state x_t, the information vector u_t and the parity vector y_t of $\mathscr{C}\mathscr{C}^{(1)}$ are given by

$$x_t = \begin{pmatrix} x_t^{(2)} \\ x_t^{(1)} \end{pmatrix}, \quad u_t = u_t^{(1)}, \quad \text{and} \quad y_t = \begin{pmatrix} y_t^{(2)} \\ y_t^{(1)} \end{pmatrix}. \tag{8}$$

So, the codewords v_t of $\mathscr{C}\mathscr{C}^{(1)}$ are given by

$$v_t = \begin{pmatrix} y_t \\ u_t \end{pmatrix} = \begin{pmatrix} y_t^{(2)} \\ y_t^{(1)} \\ u_t^{(1)} \end{pmatrix} = \begin{pmatrix} y_t^{(2)} \\ v_t^{(1)} \end{pmatrix} = \begin{pmatrix} y_t^{(2)} \\ u_t^{(2)} \end{pmatrix} = v_t^{(2)}. \tag{9}$$

Observe that a codeword of $\mathscr{C}\mathscr{C}^{(1)}$ is a codeword of $\mathscr{C}_{\mathrm{i}}$. Nevertheless, a codeword of $\mathscr{C}_{\mathrm{i}}$ is not necessarily a codeword of $\mathscr{C}\mathscr{C}^{(1)}$ (see Fig. 1).

The next theorem introduces an input-state-output representation of the concatenated convolutional code $\mathscr{C}\mathscr{C}^{(1)}$ from an input-state-output representation of the outer and inner codes. In this paper we denote by O the zero matrix of the appropriate size.

Theorem 2. *Let $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ be an (m, k, δ_1)-code and let $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$ be an (n, m, δ_2)-code. Then an input-state-output representation for the rate k/n concatenated code $\mathscr{C}\mathscr{C}^{(1)}$ is given by system* (1), *where*

$$\begin{aligned} A &= \begin{pmatrix} A_2 & B_{21}C_1 \\ O & A_1 \end{pmatrix}, \quad B = \begin{pmatrix} B_{21}D_1 + B_{22} \\ B_1 \end{pmatrix}, \\ C &= \begin{pmatrix} C_2 & D_{21}C_1 \\ O & C_1 \end{pmatrix}, \quad D = \begin{pmatrix} D_{21}D_1 + D_{22} \\ D_1 \end{pmatrix}, \end{aligned} \tag{10}$$

where $B_2 = (B_{21} \quad B_{22})$ and $D_2 = (D_{21} \quad D_{22})$, with B_{21}, B_{22}, D_{21} and D_{22} matrices of sizes $\delta_2 \times (m-k)$, $\delta_2 \times k$, $(n-m) \times (m-k)$ and $(n-m) \times k$, respectively.

Proof. From expression (1) we have, for the code $\mathscr{C}_{\mathrm{o}}$,

$$\begin{aligned} x_{t+1}^{(1)} &= A_1 x_t^{(1)} + B_1 u_t^{(1)}, \\ y_t^{(1)} &= C_1 x_t^{(1)} + D_1 u_t^{(1)}, \end{aligned}$$

and for the code $\mathscr{C}_{\mathrm{i}}$,

$$\begin{aligned} x_{t+1}^{(2)} &= A_2 x_t^{(2)} + B_2 u_t^{(2)}, \\ y_t^{(2)} &= C_2 x_t^{(2)} + D_2 u_t^{(2)}. \end{aligned}$$

Now, taking into account that $u_t^{(2)} = v_t^{(1)}$, if we consider the block partition of $B_2 = (B_{21} \quad B_{22})$ and $D_2 = (D_{21} \quad D_{22})$, in accordance with the block partition of $u_t^{(2)} = (u_{t,1}^{(2)}, u_{t,2}^{(2)})$, where $u_{t,1}^{(2)} \in \mathbb{F}^{m-k}$ and $u_{t,2}^{(2)} \in \mathbb{F}^k$, then an input-state-output representation of the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$ is given by

$$\begin{aligned} x_{t+1} &= \begin{pmatrix} A_2 & B_{21}C_1 \\ O & A_1 \end{pmatrix} x_t + \begin{pmatrix} B_{21}D_1 + B_{22} \\ B_1 \end{pmatrix} u_t, \\ y_t &= \begin{pmatrix} C_2 & D_{21}C_1 \\ O & C_1 \end{pmatrix} x_t + \begin{pmatrix} D_{21}D_1 + D_{22} \\ D_1 \end{pmatrix} u_t. \qquad \square \end{aligned}$$

By Theorems 1 and 2, it follows that the transfer function $T(z)$ associated to the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$ has the form given by Eq. (11) in terms of the transfer functions $T_1(z)$ and $T_2(z)$ associated to the outer code $\mathscr{C}_{\mathrm{o}}$ and the inner code $\mathscr{C}_{\mathrm{i}}$, respectively.

Theorem 3. *Let $T_1(z)$ be the transfer function of the outer code $\mathscr{C}_{\mathrm{o}}$ and let $T_2(z)$ be the transfer function of the inner code $\mathscr{C}_{\mathrm{i}}$. Then the transfer function $T(z)$ associated to the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$ is*

$$T(z) = \begin{pmatrix} T_{21}(z)T_1(z) + T_{22}(z) \\ T_1(z) \end{pmatrix}, \tag{11}$$

where $T_2(z) = (T_{21}(z) \quad T_{22}(z))$ with $T_{21}(z)$ and $T_{22}(z)$ matrices of sizes $(n-m) \times (m-k)$ and $(n-m) \times k$, respectively.

Now, we are interested in the conditions of the matrices A_l, B_l, C_l, and D_l, for $l = 1, 2$, of the outer and inner codes so that the concatenated code has a "good" representation. The next example shows that it is not enough for the pair (A_l, B_l) to be controllable, for $l = 1, 2$, in order to get a controllable pair (A, B) of the concatenated code.

Example 1. Let α be a primitive element of the Galois field $\mathbb{F} = GF(8)$ with $\alpha^3 + \alpha + 1 = 0$, and consider the $(2, 1, 2)$-outer code $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$, where

$$A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^6 \end{pmatrix}, \quad C_1 = (\alpha^4 \quad \alpha^3), \quad D_1 = (1 \quad \alpha^4),$$

and a (3, 2, 2)-inner code $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$, where

$$A_2 = \begin{pmatrix} \alpha^4 & 1 \\ \alpha^3 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 1 \\ \alpha & 1 & \alpha \end{pmatrix},$$

and C_2 and D_2 are arbitrary matrices.

For all $z \in \overline{\mathbb{F}}$ we have that

$$\operatorname{rank}(zI_{\delta_1} - A_1 \quad B_1) = \operatorname{rank} \begin{pmatrix} z+\alpha & 0 & 1 & 0 \\ 0 & z+\alpha^2 & 0 & \alpha^6 \end{pmatrix} = 2,$$

$$\operatorname{rank}(zI_{\delta_2} - A_2 \quad B_2) = \operatorname{rank} \begin{pmatrix} z+\alpha^4 & 1 & 1 & 0 & 1 \\ \alpha^3 & z & \alpha & 1 & \alpha \end{pmatrix} = 2$$

and therefore, the pairs (A_1, B_1) and (A_2, B_2) are controllable.

Now, from Theorem 2, the matrices A and B for an input-state-output representation of the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$ are

$$A = \begin{pmatrix} \alpha^4 & 1 & \alpha^4 & \alpha^3 \\ \alpha^3 & 0 & \alpha^5 & \alpha^4 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha^2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & \alpha^5 \\ \alpha^3 & \alpha^6 \\ 1 & 0 \\ 0 & \alpha^6 \end{pmatrix}.$$

Then, the pair (A, B) is not controllable because

$$\operatorname{rank}(\alpha I - A \quad B) = \operatorname{rank} \begin{pmatrix} \alpha^2 & 1 & \alpha^4 & \alpha^3 & 1 & \alpha^5 \\ \alpha^3 & \alpha & \alpha^5 & \alpha^4 & \alpha^3 & \alpha^6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha^4 & 0 & \alpha^6 \end{pmatrix} = 3 \neq 4.$$

The next theorem gives conditions which ensure both the controllability of the pair (A, B) and the observability of the pair (A, C).

Theorem 4. *Let $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ be an (m, k, δ_1)-code and let $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$ be an (n, m, δ_2)-code. Let $\mathscr{C}\mathscr{C}^{(1)}(A, B, C, D)$ be the concatenated code described by the matrices in expression* (10).

(a) *If* $\operatorname{rank}(B) = \delta_1 + \delta_2$, *then (A, B, C, D) is a minimal representation of $\mathscr{C}\mathscr{C}^{(1)}$ with complexity $\delta_1 + \delta_2$.*

(b) *If the pair (A_l, C_l) is observable for $l = 1, 2$, then the pair (A, C) is observable.*

Proof. (a) Since $\operatorname{rank}(B) = \delta_1 + \delta_2$, it is clear that

$$\operatorname{rank}(zI - A \quad B) = \delta_1 + \delta_2 \quad \text{for all } z \in \overline{\mathbb{F}}.$$

So, the pair (A, B) is controllable and, consequently, (A, B, C, D) is a minimal representation of $\mathscr{C}\mathscr{C}^{(1)}$.

(b) For all $z \in \overline{\mathbb{F}}$, we have, from the size of the matrix $\begin{pmatrix} zI - A \\ C \end{pmatrix}$, that

$$\operatorname{rank} \begin{pmatrix} zI - A \\ C \end{pmatrix} \leqslant \delta_1 + \delta_2.$$

Furthermore,

$$\begin{aligned}\operatorname{rank}\begin{pmatrix} zI - A \\ C \end{pmatrix} &= \operatorname{rank}\begin{pmatrix} zI_{\delta_2} - A_2 & -B_{21}C_1 \\ C_2 & D_{21}C_1 \\ O & zI_{\delta_1} - A_1 \\ O & C_1 \end{pmatrix} \\ &\geqslant \operatorname{rank}\begin{pmatrix} zI_{\delta_2} - A_2 \\ C_2 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} zI_{\delta_1} - A_1 \\ C_1 \end{pmatrix} \\ &= \delta_2 + \delta_1.\end{aligned}$$

So the rank$\begin{pmatrix} zI - A \\ C \end{pmatrix} = \delta_1 + \delta_2$, and the pair (A, C) is observable. □

Now, as a consequence of Theorem 4 and Lemma 1, we have the following result.

Corollary 1. *Let $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ be an (m, k, δ_1)-code and let $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$ be an (n, m, δ_2)-code. Let $\mathscr{C}\mathscr{C}^{(1)}(A, B, C, D)$ be the concatenated code described by the matrices in expression (10). Assume the following two conditions hold*:

(a) $\operatorname{Rank}(B) = \delta_1 + \delta_2$.
(b) *The matrices (A_l, C_l) form an observable pair, for $l = 1, 2$.*

Then (A, B, C, D) is a minimal representation of the observable convolutional code $\mathscr{C}\mathscr{C}^{(1)}$ with complexity $\delta_1 + \delta_2$.

Remark 2. Note that the condition (a) of Theorem 4 implies that the pair (A_l, B_l) is controllable, for $l = 1, 2$.

Firstly, from (a) and the sizes of B_1 and B_2, it follows that

$$\operatorname{rank}(B_{21}D_1 + B_{22}) = \delta_2, \tag{12}$$

$$\operatorname{rank}(B_1) = \delta_1. \tag{13}$$

Now, from expression (13), we obtain that B_1 has full row rank. So the matrices (A_1, B_1) form a controllable pair.

On the other hand, if (A_2, B_2) is not a controllable pair, then, from relation (2.14) of [26], the matrix B_2 can be expressed as $B_2 = (B_{21} \quad B_{22}) = S^{-1}\begin{pmatrix} \widetilde{B}_2 \\ O \end{pmatrix}$, so it is not fullrank. Furthermore, if $\operatorname{rank}(B_2) = r_2 < \delta_2$, then we obtain that

$$\operatorname{rank}(B_2 D_1 + B_{22}) \leqslant r_2 < \delta_2,$$

which contradicts (12).

Remark 3. Example 1 also shows that the converse of Remark 2 is not true in general because the pair (A_l, B_l) is controllable, for $l = 1, 2$, but $\operatorname{rank}(B) = 2 \neq 2 + 2$.

In the following example, we consider a rate 2/3 convolutional code as outer code and a rate 3/4 convolutional code as inner code. Then, we apply Corollary 1 in order to get a minimal representation of the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$.

Example 2. Let α be a primitive element of the field $\mathbb{F} = GF(4)$, with $\alpha^2 + \alpha + 1 = 0$. Let $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ be the (3, 2, 1)-code, where

$$A_1 = (\alpha^2), \quad B_1 = (1 \quad \alpha), \quad C_1 = (1), \quad D_1 = (\alpha^2 \quad \alpha).$$

Let $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$ be the (4, 3, 1)-code, where

$$A_2 = (1), \quad B_2 = (1 \quad \alpha \quad \alpha^2), \quad C_2 = (1), \quad D_2 = (1 \quad 1 \quad 1).$$

By applying Theorem 2 it follows then that the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & \alpha^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & \alpha \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} \alpha & \alpha^2 \\ \alpha^2 & \alpha \end{pmatrix}$$

are a representation of the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$. Furthermore, since $\operatorname{rank}(B) = 2 = \delta_1 + \delta_2$, through part (a) of Theorem 4, the above matrices are a minimal description for $\mathscr{C}\mathscr{C}^{(1)}$. In addition, since the pairs (A_1, C_1) and (A_2, C_2) are observable, we have by part (b) of Theorem 4 that (A, C) forms an observable pair. Finally Corollary 1 ensures us that $\mathscr{C}\mathscr{C}^{(1)}$ is an observable convolutional code.

The next result gives conditions in order to achieve the controllability of the pair (A, B) of the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$, for the particular case where the outer code has rate $1/2$ and complexity $\delta_1 = 1$ and the matrix A_2 is a diagonal matrix.

Theorem 5. *Let $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ be a (2, 1, 1)-code and let $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$ be an $(n, 2, \delta_2)$-code. Let $\mathscr{C}\mathscr{C}^{(1)}(A, B, C, D)$ the concatenated code described by the matrices in expression (10). Assume the following conditions hold:*

(a) *A_2 is a diagonal matrix.*
(b) *The matrices (A_1, B_1) form a controllable pair and the matrices (A_2, B_{21}) form a controllable pair.*
(c) *All the elements of vector B_{21} are nonzero and B_{22} is a zero vector.*
(d) $\operatorname{rank}\begin{pmatrix} -C_1 & D_1 \\ \lambda I - A_1 & B_1 \end{pmatrix} = 2$ *for all λ eigenvalue of A.*

Then, (A, B, C, D) is a minimal representation of the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$ with complexity $\delta_1 + \delta_2 = 1 + \delta_2$.

Proof. Taking into account that the outer code $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ is a (2, 1, 1)-code, the matrices $A_1 = (a_1), B_1 = (b_1), C_1 = (c_1)$, and $D_1 = (d_1)$ are scalar matrices. Now, since B_{21} is a $(\delta_2 \times 1)$-vector and the matrices (A_2, B_{21}) form a controllable pair, then all the eigenvalues of A_2 are different. Let

$$A_2 = \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_{\delta_2}] \quad \text{and} \quad B_{21} = \begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{\delta_2 2} \end{pmatrix}.$$

First, assume that λ is an eigenvalue of A with $\lambda \neq a_1$. We can assume, without loss of generality, that $\lambda = \lambda_1$. Then,

$$(\lambda_1 I - A \quad B) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -b_{12}c_1 & b_{12}d_1 \\ 0 & \lambda_1 - \lambda_2 & \cdots & 0 & -b_{22}c_1 & b_{22}d_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_1 - \lambda_{\delta_2} & -b_{\delta_2 2}c_1 & b_{\delta_2 2}d_1 \\ 0 & 0 & \cdots & 0 & \lambda_1 - a_1 & b_1 \end{pmatrix},$$

and taking into account that all the eigenvalues of A_2 are different and conditions (c) and (d), we get

$$\operatorname{rank}(\operatorname{diag}[\lambda_1 - \lambda_2, \lambda_1 - \lambda_3, \ldots, \lambda_1 - \lambda_{\delta_2}]) = \delta_2 - 1$$

and

$$\operatorname{rank}\begin{pmatrix} -b_{12}c_1 & b_{12}d_1 \\ \lambda_1 - a_1 & b_1 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -C_1 & D_1 \\ \lambda_1 I - A_1 & B_1 \end{pmatrix} = 2.$$

Then,

$$\operatorname{rank}(\lambda_1 I - A \quad B) = 2 + \delta_2 - 1 = 1 + \delta_2 = \delta_1 + \delta_2.$$

Let λ an eigenvalue of A with $\lambda = a_1$. If λ is not an eigenvalue of A_2, and we denote

$$\widetilde{A}_2 = \operatorname{diag}[\lambda - \lambda_1, \lambda - \lambda_2, \ldots, \lambda - \lambda_{\delta_2}] = \operatorname{diag}[a_1 - \lambda_1, a_1 - \lambda_2, \ldots, a_1 - \lambda_{\delta_2}]$$

then, $\operatorname{rank}(\widetilde{A}_2) = \delta_2$, so

$$\operatorname{rank}(\lambda I - A \quad B) = \operatorname{rank}(a_1 I - A \quad B) = \operatorname{rank}\begin{pmatrix} \widetilde{A}_2 & -B_2 C_1 & B_2 D_1 \\ O & O & B_1 \end{pmatrix} = \delta_2 + 1,$$

since from condition (b), the pair (A_1, B_1) is controllable and, in particular, $b_1 \neq 0$, because $\delta_1 = 1$.

Finally, if $\lambda = a_1$ is also an eigenvalue of A_2, then we can assume, without loss of generality, that $\lambda = \lambda_1$. Then, taking into account that all the eigenvalues of A_2 are different,

$$\operatorname{rank}(\lambda I - A \quad B) = \operatorname{rank}\begin{pmatrix} 0 & 0 & \cdots & 0 & -b_{12}c_1 & b_{12}d_1 \\ 0 & \lambda_1 - \lambda_2 & \cdots & 0 & -b_{22}c_1 & b_{22}d_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_1 - \lambda_{\delta_2} & -b_{\delta_2 2}c_1 & b_{\delta_2 2}d_1 \\ 0 & 0 & \cdots & 0 & 0 & b_1 \end{pmatrix} = \delta_2 + 1$$

since from conditions (c) and (d),

$$\operatorname{rank}\begin{pmatrix} -b_{12}c_1 & b_{12}d_1 \\ 0 & b_1 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -b_{12}c_1 & b_{12}d_1 \\ \lambda - a_1 & b_1 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -C_1 & D_1 \\ \lambda I - A_1 & B_1 \end{pmatrix} = 2.$$

We conclude then,

$$\operatorname{rank}(zI - A \quad B) = \delta_1 + \delta_2 \quad \text{for all } z \in \overline{\mathbb{F}},$$

that is, the matrices (A, B) form a controllable pair. $\square$

Remark 4. Using the conditions of the above theorem, $\lambda = a_1$ is an eigenvalue of matrix A, so condition (d) is simplified to

$$\operatorname{rank}\begin{pmatrix} -C_1 & D_1 \\ \lambda I - A_1 & B_1 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -C_1 & D_1 \\ O & B_1 \end{pmatrix} = 2,$$

and consequently the matrices (A_1, C_1) form an observable pair, since $\delta_1 = 1$.

For the particular case where the complexity of the outer code and the inner code is $\delta_1 = \delta_2 = 1$, we get the following result.

Theorem 6. *Let $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ be an $(m, k, 1)$-code and let $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$ be an $(n, m, 1)$-code. Let $\mathscr{CC}^{(1)}(A, B, C, D)$ be the concatenated code described by matrices in expression (10). Assume the following conditions hold:*

(a) *The matrices (A_1, B_1) form a controllable pair.*
(b) *The matrix $\begin{pmatrix} -B_{21}C_1 & B_{21}D_1 + B_{22} \\ A_2 - A_1 & B_1 \end{pmatrix}$, of size $2 \times (k + 1)$, has rank equal to 2.*

Then, (A, B, C, D) is a minimal representation of the concatenated code $\mathscr{CC}^{(1)}(A, B, C, D)$ with complexity $\delta_1 + \delta_2 = 2$.

Proof. First, note that since $\delta_1 = \delta_2 = 1$, we have that $A_1 = (a_1)$ and $A_2 = (a_2)$. So, the eigenvalues of the matrix A of the code $\mathscr{CC}^{(1)}(A, B, C, D)$ are a_1 and a_2. Supposing that $a_1 = a_2 = \lambda$, then, from condition (b), we get

$$\operatorname{rank}(\lambda I - A \quad B) = \operatorname{rank}\begin{pmatrix} O & -B_{21}C_1 & B_{21}D_1 + B_{22} \\ O & O & B_1 \end{pmatrix} = 2.$$

So $\operatorname{rank}(zI - A \quad B) = 2 = \delta_1 + \delta_2$ for all $z \in \overline{\mathbb{F}}$, that is, the pair (A, B) is controllable. Now, suppose that $a_1 \neq a_2$. Then,

$$2 \geqslant \operatorname{rank}(a_1 I - A \quad B) = \operatorname{rank}\begin{pmatrix} a_1 - a_2 & -B_{21}C_1 & B_{21}D_1 + B_{22} \\ 0 & 0 & B_1 \end{pmatrix}$$

$$\geqslant \operatorname{rank}\begin{pmatrix} a_1 - a_2 & B_2 D_1 \\ 0 & B_1 \end{pmatrix} = 2$$

since from condition (a), we get $B_1 \neq 0$. Furthermore, from condition (b),

$$\operatorname{rank}(a_2 I - A \quad B) = \operatorname{rank}\begin{pmatrix} 0 & -B_{21}C_1 & B_{21}D_1 + B_{22} \\ 0 & a_2 - a_1 & B_1 \end{pmatrix} = 2.$$

Then, $\operatorname{rank}(zI - A \quad B) = 2 = \delta_1 + \delta_2$ for all $z \in \overline{\mathbb{F}}$, that is, the pair (A, B) is controllable. $\square$

Observe that the condition

$$\operatorname{rank}\begin{pmatrix} -B_{21}C_1 & B_{21}D_1 + B_{22} \\ A_2 - A_1 & B_1 \end{pmatrix} = 2$$

of Theorem 6, does not imply necessarily that (A_2, B_{21}) is controllable. In fact, only if the matrix B_{22} is the zero matrix, the previous condition implies the controllability of (A_2, B_{21}). In the rest of the cases, we do not have this result, as we can see in the next example.

Example 3. As in Example 1, let α be a primitive element of $\mathbb{F} = GF(8)$. Let $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ be a (4, 2, 1)-code, where C_1 and D_1 are arbitrary matrices of sizes 2×1 and 2×2, respectively, and

$$A_1 = (\alpha), \quad B_1 = (1 \quad 1).$$

Let $\mathscr{C}_\text{i}(A_2, B_2, C_2, D_2)$ be a (5, 4, 1)-code, where C_2 and D_2 are arbitrary matrices of sizes 1×1 and 1×4, respectively, and

$$A_2 = (\alpha^3), \quad B_2 = (B_{21} \quad B_{22}) = (0 \quad 0 \quad \alpha \quad \alpha).$$

Observe that the pair (A_2, B_{21}) is not controllable, since $B_{21} = (0 \quad 0)$. Nevertheless,

$$\operatorname{rank}\begin{pmatrix} -B_{21}C_1 & B_{21}D_1 + B_{22} \\ A_2 - A_1 & B_1 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & \alpha & \alpha \\ 1 & 1 & 1 \end{pmatrix} = 2,$$

so the condition (b) of Theorem 6 holds.

Observe also that the pair (A_1, B_1) is controllable, so the condition (a) of Theorem 6 also holds. So we can conclude that the pair (A, B) of the code $\mathscr{C}\mathscr{C}^{(1)}(A, B, C, D)$ is controllable, where

$$A = \begin{pmatrix} \alpha^3 & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha & \alpha \\ 1 & 1 \end{pmatrix}.$$

Remark 5. The previous example shows that the condition (b) of Theorem 6 does not imply that the pair (A_2, B_{21}) is controllable, but that condition implies that the pair (A_2, B_2) is controllable, since $\delta_2 = 1$.

Remark 6. If in Theorem 6 we do not require the pair (A_1, B_1) to be controllable, then the matrix B_1 is the zero matrix, so

$$\operatorname{rank}(a_1 I - A_2 \quad B) = \operatorname{rank}\begin{pmatrix} A_1 - A_2 & -B_{21}C_1 & B_{21}D_1 + B_{22} \\ O & O & O \end{pmatrix} \leqslant \delta_2 < 2$$

and (A, B) is not a controllable pair.

If in Theorem 6, we do not require condition (b), pair (A, B) can be a non controllable pair, as we can see in the following example.

Example 4. As in Example 1, let α be a primitive element of $\mathbb{F} = GF(8)$ and let $\mathscr{C}_\text{o}(A_1, B_1, C_1, D_1)$ be the (2, 1, 1)-code described by the matrices

$$A_1 = (1), \quad B_1 = (\alpha^2), \quad C_1 = (\alpha^6), \quad D_1 = (1).$$

Then, the pair (A_1, B_1) is controllable. Now, let $\mathscr{C}_\text{i}(A_2, B_2, C_2, D_2)$ be an $(n, 2, 1)$-code, with

$$A_2 = (\alpha^3) \quad \text{and} \quad B_2 = (\alpha \quad 0),$$

where C_2 and D_2 are arbitrary matrices of sizes $(n-2) \times 1$ and $(n-2) \times 2$, respectively, so that the pair (A_2, C_2) is observable (for example, we can consider $C_2 = (1)$). Then, taking into account Theorem 2, the matrices A and B of the concatenated code $\mathscr{C}\mathscr{C}^{(1)}(A, B, C, D)$ are given by

$$A = \begin{pmatrix} \alpha^3 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha \\ \alpha^2 \end{pmatrix}.$$

Note that

$$\operatorname{rank}\begin{pmatrix} -B_{21}C_1 & B_{21}D_1 + B_{22} \\ A_2 - A_1 & B_1 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{pmatrix} = 1 < 2,$$

so the condition (b) of Theorem 6 does not verify. Furthermore,

$$\operatorname{rank}(\alpha^3 I - A \quad B) = \operatorname{rank}\begin{pmatrix} 0 & 1 & \alpha \\ 0 & \alpha & \alpha^2 \end{pmatrix} = 1 < 2 = \delta_1 + \delta_2$$

so (A, B) is not a controllable pair.

As a consequence of Theorem 6, we get the following result for the particular case where the matrices A_1 and A_2 are equal.

Corollary 2. *Let $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ be an $(m, k, 1)$-code, and let $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$ be an $(n, m, 1)$-code. Let $\mathscr{C}\mathscr{C}^{(1)}(A, B, C, D)$ be the concatenated code described by the matrices in expression (10). Assume that the following conditions hold:*

(a) $A_1 = A_2$.
(b) *The matrices (A_1, B_1) form a controllable pair.*
(c) *The vectors B_{21} and C_1 are not orthogonal vectors.*

Then, (A, B, C, D) is a minimal representation for the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$ with complexity $\delta_1 + \delta_2 = 2$.

Remark 7. Note that condition (c) of Corollary 2 implies in particular that the matrices (A_2, B_{21}) form a controllable pair (and then, the matrices (A_2, B_2) form a controllable pair, since $\delta_2 = 1$) and the matrices (A_1, C_1) form an observable pair.

If the outer code has rate $k/(k + 1)$, then the matrices B_{21} and C_1 are of size 1×1. Then, the vectors B_{21} and C_1 are orthogonal if and only if one of them is the zero vector; so it is sufficient to have controllability of pair (A_2, B_{21}) and observability of (A_1, C_1) so that the condition (c) of Corollary 2 holds. Then, we get the following result.

Corollary 3. *Let $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ be a $(k + 1, k, 1)$-code and let $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$ be an $(n, k + 1, 1)$-code. Let $\mathscr{C}\mathscr{C}^{(1)}(A, B, C, D)$ be the concatenated code described by matrices in expression (10). Assume the following conditions hold:*

(a) $A_1 = A_2$.
(b) *The matrices (A_1, B_1) form a controllable pair.*
(c) *The matrices (A_2, B_{21}) form a controllable pair.*
(d) *The matrices (A_1, C_1) form an observable pair.*

Then, (A, B, C, D) is a minimal representation of the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$ with complexity $\delta_1 + \delta_2 = 2$.

Example 1 shows that it is not enough for the pair (A_l, B_l) to be controllable, for $l = 1, 2$, in order to get a controllable pair (A, B) of the concatenated code. For the particular case where the outer code has rate $1/2$ and all eigenvalues of A_2 are eigenvalues of A_1, that is, $\sigma(A_2) \subseteq \sigma(A_1)$, where $\sigma(A_i)$ is the spectrum of A_i, we have the following result.

Theorem 7. *Let $\mathscr{C}_o(A_1, B_1, C_1, D_1)$ be a $(2, 1, \delta_1)$-code and let $\mathscr{C}_i(A_2, B_2, C_2, D_2)$ be an $(n, 2, \delta_2)$-code. Let $\mathscr{CC}^{(1)}(A, B, C, D)$ be the concatenated code described by the matrices in expression* (10). *Assume the following conditions hold*:

(a) $\sigma(A_2) \subseteq \sigma(A_1)$.
(b) *The matrices (A_1, B_1) form a controllable pair.*
(c) *The matrices (A_2, B_{21}) form a controllable pair.*
(d) *All the elements of matrix B_{21} are nonzero.*
(e) *The matrices (A_l, C_l) form an observable pair, for $l = 1, 2$.*

Then, (A, B, C, D) is a minimal representation of the concatenated code $\mathscr{CC}^{(1)}$ with complexity $\delta_1 + \delta_2$. Furthermore, $\mathscr{CC}^{(1)}$ is an observable convolutional code.

Proof. Firstly, assume that $\delta_1 = 1$, then from condition (e), the pair (A_1, C_1) is observable, so the (1×1)-matrix C_1 must be nonzero. Now, taking into account conditions (a) and (c),

$$\operatorname{rank}(zI - A \quad B) = 1 + \delta_2 = \delta_1 + \delta_2$$

for all $z \in \overline{\mathbb{F}}$.

Now, assume that $\delta_1 > 1$ and suppose that for some $z \in \overline{\mathbb{F}}$, we have

$$\operatorname{rank}(zI - A \quad B) < \delta_1 + \delta_2, \tag{14}$$

so from condition (a), $z \in \sigma(A) = \sigma(A_1)$. By condition (b),

$$\delta_1 = \operatorname{rank}(zI_{\delta_1} - A_1 \quad B_1) = \operatorname{rank}(O \quad zI_{\delta_1} - A_1 \quad B_1) \tag{15}$$

and, by conditions (c), (d) and (e), it follows that

$$\delta_2 = \operatorname{rank}(zI_{\delta_2} - A_2 \quad B_{21}) = \operatorname{rank}(zI_{\delta_2} - A_2 \quad -B_{21}C_1). \tag{16}$$

Now, taking into account the relations (14)–(16), we have that one row of the matrix $(zI_{\delta_2} - A_2 \quad -B_{21}C_1 \quad B_{21}D_1 + B_{22})$ is a linear combination of the rows of the matrix $(O \quad zI_{\delta_1} - A_1 \quad B_1)$ or one row of the matrix $(O \quad zI_{\delta_1} - A_1 \quad B_1)$ is a linear combination of the rows of the matrix $(zI_{\delta_2} - A_2 \quad -B_{21}C_1 \quad B_{21}D_1 + B_{22})$. But then, from condition (d), $\operatorname{rank}\begin{pmatrix} zI_{\delta_1} - A_1 \\ C_1 \end{pmatrix} < \delta_1$, which contradicts condition (e) for $l = 1$.

The observability condition for (A, C) follows from condition (e) using a similar argument to Theorem 4. Finally, we get the observability condition of $\mathscr{CC}^{(1)}$ by using Lemma 1. $\square$

If in Theorem 7 we do not require the controllability condition of (A_2, B_{21}) then, the representation of the concatenated code $\mathscr{CC}^{(1)}$ may be nonminimal, as we can see in the following example.

Example 5. For matrices in Example 1, the pair (A_2, B_2) is controllable. Nevertheless, the pair $(A_2, B_{21}) = \left(\begin{pmatrix} \alpha^4 & 1 \\ \alpha^3 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \alpha \end{pmatrix}\right)$ is not controllable because

$$\operatorname{rank}(\alpha I - A_2 \quad B_{21}) = \operatorname{rank}\begin{pmatrix} \alpha^2 & 1 & 1 \\ \alpha^3 & \alpha & \alpha \end{pmatrix} = 1 \neq 2.$$

Furthermore, $\sigma(A_2) = \{\alpha, \alpha^2\} = \sigma(A_1)$, the matrices (A_1, B_1) and (A_1, C_1) form a controllable and an observable pair, respectively, and all the elements of B_{21} are nonzero. So all the conditions but condition (c) are verified. Now, since the pair (A, B) is not controllable (see Example 1), the representation (A, B, C, D) is nonminimal.

Theorem 7 does not verify if $\sigma(A_2) \nsubseteq \sigma(A_1)$, as we can see in the following example.

Example 6. Let α be a primitive element of $\mathbb{F} = GF(8)$, as in Example 1. Let $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ be the (2, 1, 1)-code described by matrices

$$A_1 = (\alpha^2), \quad B_1 = (\alpha), \quad C_1 = (1), \quad D_1 = (\alpha)$$

and let $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$ be a (3, 2, 1)-code described by matrices

$$A_2 = \begin{pmatrix} \alpha^2 & 0 \\ 1 & \alpha^6 \end{pmatrix}, \quad B_2 = (B_{21} \quad B_{22}) = \begin{pmatrix} \alpha & 0 \\ 1 & 0 \end{pmatrix},$$

where C_2 and D_2 are arbitrary matrices of sizes 1×2 and 1×1, respectively, so that (A_2, C_2) is observable. Observe that (A_1, B_1) and (A_2, B_{21}) are controllable pairs, (A_l, C_l) is observable, for $l = 1, 2$ and all the elements of B_{21} are nonzero. Nevertheless,

$$\sigma(A_2) = \{\alpha^2, \alpha^6\} \nsubseteq \{\alpha^2\} = \sigma(A_1).$$

Taking into account Theorem 2, the matrices A and B of an input-state-output representation of the code $\mathscr{C}\mathscr{C}^{(1)}$ are given by

$$A = \begin{pmatrix} \alpha^2 & 0 & \alpha \\ 1 & \alpha^6 & 1 \\ 0 & 0 & \alpha^2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha^2 \\ \alpha \\ \alpha \end{pmatrix}.$$

Now,

$$\operatorname{rank}(\alpha^6 I - A \quad B) = \operatorname{rank} \begin{pmatrix} 1 & 0 & \alpha & \alpha^2 \\ 1 & 0 & 1 & \alpha \\ 0 & 0 & 1 & \alpha \end{pmatrix} = 2 \neq 3,$$

so (A, B) is not a controllable pair.

Once we have seen that we can obtain "good" representations for the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$, we provide, in the next theorem, a lower bound for the free distance of $\mathscr{C}\mathscr{C}^{(1)}$ in terms of the free distances of $\mathscr{C}_{\mathrm{o}}$ and $\mathscr{C}_{\mathrm{i}}$. Firstly, we obtain a lower bound for the column distances of $\mathscr{C}\mathscr{C}^{(1)}$, also in terms of the column distances of $\mathscr{C}_{\mathrm{o}}$ and $\mathscr{C}_{\mathrm{i}}$.

Lemma 2. *Let $\mathscr{C}\mathscr{C}^{(1)}$ be the concatenated code given by Theorem 2 from the outer code $\mathscr{C}_{\mathrm{o}}$ and the inner code $\mathscr{C}_{\mathrm{i}}$. Then*

$$d_j^c(\mathscr{C}\mathscr{C}^{(1)}) \geqslant \max\{d_j^c(\mathscr{C}_{\mathrm{o}}), d_j^c(\mathscr{C}_{\mathrm{i}})\} \quad \textit{for } j = 0, 1, 2, \ldots \tag{17}$$

Proof. Taking into account expression (5), the relations between y_t, $y_t^{(1)}$, $y_t^{(2)}$; u_t, $u_t^{(1)}$, $u_t^{(2)}$, and v_t, $v_t^{(1)}$, $v_t^{(2)}$ given by expressions (6)–(9), and that the condition $u_0 \neq 0$ implies $u_0^{(1)} \neq 0$ and $u_0^{(2)} \neq 0$, we have

$$d_j^c(\mathscr{C}\mathscr{C}^{(1)}) = \min_{u_0 \neq 0} \left\{ \sum_{t=0}^{j} \mathrm{wt}(v_t) \right\} \geqslant \min_{u_0^{(1)} \neq 0} \left\{ \sum_{t=0}^{j} \mathrm{wt}\left(v_t^{(1)}\right) \right\} = d_j^c(\mathscr{C}_{\mathrm{o}}), \tag{18}$$

$$d_j^c(\mathscr{C}\mathscr{C}^{(1)}) = \min_{u_0 \neq 0} \left\{ \sum_{t=0}^{j} \mathrm{wt}(v_t) \right\} \geqslant \min_{u_0^{(2)} \neq 0} \left\{ \sum_{t=0}^{j} \mathrm{wt}\left(v_t^{(2)}\right) \right\} = d_j^c(\mathscr{C}_\mathrm{i}). \tag{19}$$

Now, inequality (17) follows from inequalities (18) and (19). □

Now, as an immediate consequence of expression (4) and the above lemma we obtain the following result.

Theorem 8. *Let $\mathscr{C}\mathscr{C}^{(1)}$ be the concatenated code given by Theorem 2 from the outer code $\mathscr{C}_\mathrm{o}$ and the inner code $\mathscr{C}_\mathrm{i}$. Then,*

$$d_{\mathrm{free}}(\mathscr{C}\mathscr{C}^{(1)}) \geqslant \max\{d_{\mathrm{free}}(\mathscr{C}_\mathrm{o}), d_{\mathrm{free}}(\mathscr{C}_\mathrm{i})\}. \tag{20}$$

We finish this section with three examples. In the first one, we use the construction proposed by Smarandache and Rosenthal [28] in order to consider an outer MDS convolutional code. In this case, we obtain a concatenated code $\mathscr{C}\mathscr{C}^{(1)}$ with free distance close to the Singleton bound. We use a computer algebra program to obtain the free distances.

Example 7. Let α be a primitive element of $\mathbb{F} = GF(8)$, as in Example 1. Let $\mathscr{C}_\mathrm{o}(A_1, B_1, C_1, D_1)$ be the (2, 1, 2)-code, where

$$A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C_1 = (\alpha^5 \quad \alpha^2), \quad D_1 = (1).$$

It follows that $\mathscr{C}_\mathrm{o}$ is an MDS convolutional code (see [28]), so $d_{\mathrm{free}}(\mathscr{C}_\mathrm{o}) = 6$. In addition, $\mathscr{C}_\mathrm{o}$ in an observable convolutional code and the pair (A_1, B_1) is controllable.

Let $\mathscr{C}_\mathrm{i}(A_2, B_2, C_2, D_2)$ be the (3, 2, 1)-code where

$$A_2 = (\alpha^2), \quad B_2 = (1 \quad 1), \quad C_2 = (1), \quad D_2 = (1 \quad 1).$$

Through computation, we get $d_{\mathrm{free}}(\mathscr{C}_\mathrm{i}) = 2$, the pair (A_2, B_{21}) is controllable, all the elements of the matrix B_{21} are nonzero, and the pair (A_2, C_2) is observable.

Furthermore,

$$\sigma(A_2) = \{\alpha^2\} \subseteq \{\alpha, \alpha^2\} = \sigma(A_1).$$

So, by applying Theorem 2 and Theorem 7, a minimal representation of the observable code $\mathscr{C}\mathscr{C}^{(1)}$ is given by the matrices

$$A = \begin{pmatrix} \alpha^2 & \alpha^5 & \alpha^2 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & \alpha^5 & \alpha^2 \\ 0 & \alpha^5 & \alpha^2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now, by using Theorem 8 and the Singleton bound we have

$$6 = \max\{6, 2\} \leqslant d_{\mathrm{free}}(\mathscr{C}\mathscr{C}^{(1)}) \leqslant 12.$$

In this case $d_{\mathrm{free}}(\mathscr{C}\mathscr{C}^{(1)}) = 11$ which is close to the Singleton bound.

In the following example, we also consider an MDS convolutional code as the outer code $\mathscr{C}_\mathrm{o}$.

Example 8. As in Example 1, let α be a primitive element of the field $\mathbb{F} = GF(8)$. Consider the (2, 1, 1)-outer code $\mathscr{C}_\mathrm{o}(A_1, B_1, C_1, D_1)$, where

$$A_1 = (\alpha), \quad B_1 = (1), \quad C_1 = (\alpha^4), \quad D_1 = (1).$$

Then the pair (A_1, B_1) is controllable and the pair (A_1, C_1) is observable. Since $\mathscr{C}_{\mathrm{o}}$ is an MDS code (see [28]), we have $d_{\mathrm{free}}(\mathscr{C}_{\mathrm{o}}) = 4$.

As the inner code we consider the $(3, 2, 1)$-code $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$, where

$$A_2 = (\alpha^2), \quad B_2 = (1 \quad 1), \quad C_2 = (1), \quad D_2 = (1 \quad 1).$$

Then, the pair (A_2, B_2) is controllable and the pair (A_2, C_2) is observable. Through computation, we get $d_{\mathrm{free}}(\mathscr{C}_{\mathrm{i}}) = 2$.

Furthermore,

$$\operatorname{rank}\begin{pmatrix} -B_{21}C_1 & B_{21}D_1 + B_{22} \\ A_2 - A_1 & B_1 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \alpha^4 & 0 \\ \alpha^4 & 1 \end{pmatrix} = 2.$$

So, by Theorem 2, Theorem 6 and part (b) of Theorem 4, a minimal representation of the observable code $\mathscr{C}\mathscr{C}^{(1)}$ is given by matrices

$$A = \begin{pmatrix} \alpha^2 & \alpha^4 \\ 0 & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & \alpha^4 \\ 0 & \alpha^4 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

On the other hand, by Theorem 8 and the Singleton bound we have

$$4 = \max\{4, 2\} \leqslant d_{\mathrm{free}}(\mathscr{C}\mathscr{C}^{(1)}) \leqslant 9$$

but, in fact, $d_{\mathrm{free}}(\mathscr{C}\mathscr{C}^{(1)}) = 7$.

In the next example we consider two MDS convolutional codes as the outer and inner code, respectively.

Example 9. Let $\mathscr{C}_{\mathrm{o}}$ and $\mathscr{C}_{\mathrm{i}}$ be the outer and the inner codes of Example 2. Since both codes are MDS, we have that $d_{\mathrm{free}}(\mathscr{C}_{\mathrm{o}}) = 3$ and $d_{\mathrm{free}}(\mathscr{C}_{\mathrm{i}}) = 3$. Then, by applying Theorem 8 and the Singleton bound we have

$$3 = \max\{3, 3\} \leqslant d_{\mathrm{free}}(\mathscr{C}\mathscr{C}^{(1)}) \leqslant 7$$

but, in fact, $d_{\mathrm{free}}(\mathscr{C}\mathscr{C}^{(1)}) = 6$, which is close to the Singleton bound.

4. The second model of concatenated convolutional codes

In this section, we introduce the second model of concatenated codes. As in the previous section, let $\mathscr{C}_{\mathrm{o}}$ be the (m, k, δ_1)-outer code and $\mathscr{C}_{\mathrm{i}}$ be the $(n-k, m-k, \delta_2)$-inner code. Let $x_t^{(1)}$, $u_t^{(1)}$, and $y_t^{(1)}$ be the state vector, the information vector, and the parity vector of $\mathscr{C}_{\mathrm{o}}$, respectively. Also, let $x_t^{(2)}$, $u_t^{(2)}$, and $y_t^{(2)}$ be the state vector, the information vector, and the parity vector of $\mathscr{C}_{\mathrm{i}}$, respectively. In this model the inner code encodes only the parity vector $y_t^{(1)}$ of the codeword

$$v_t^{(1)} = \begin{pmatrix} y_t^{(1)} \\ u_t^{(1)} \end{pmatrix} \tag{21}$$

of the outer code (see Fig. 2), that is,

$$u_t^{(2)} = y_t^{(1)}. \tag{22}$$

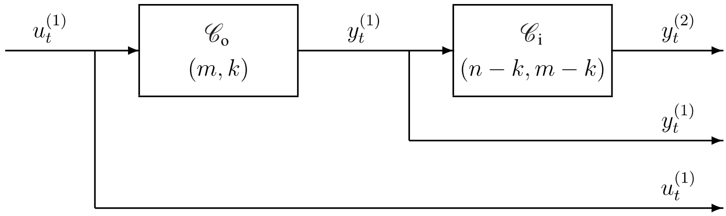

Fig. 2. Concatenated code $\mathscr{C}\mathscr{C}^{(2)}$.

We denote by $\mathscr{C}\mathscr{C}^{(2)}$ the corresponding concatenated convolutional code. Observe that the vector state x_t, the information vector u_t and the parity vector y_t of $\mathscr{C}\mathscr{C}^{(2)}$ are given, as in the previous case, by

$$x_t = \begin{pmatrix} x_t^{(2)} \\ x_t^{(1)} \end{pmatrix}, \quad u_t = u_t^{(1)}, \quad \text{and} \quad y_t = \begin{pmatrix} y_t^{(2)} \\ y_t^{(1)} \end{pmatrix}. \tag{23}$$

But in this case, the codewords v_t of $\mathscr{C}\mathscr{C}^{(2)}$ are given by

$$v_t = \begin{pmatrix} y_t \\ u_t \end{pmatrix} = \begin{pmatrix} y_t^{(2)} \\ y_t^{(1)} \\ u_t^{(1)} \end{pmatrix} = \begin{pmatrix} y_t^{(2)} \\ v_t^{(1)} \end{pmatrix}. \tag{24}$$

Nevertheless, since the codewords of $\mathscr{C}_\mathrm{i}$ are given by

$$v_t^{(2)} = \begin{pmatrix} y_t^{(2)} \\ u_t^{(2)} \end{pmatrix} = \begin{pmatrix} y_t^{(2)} \\ y_t^{(1)} \end{pmatrix}, \tag{25}$$

we have that $v_t \neq v_t^{(2)}$.

Note that the form of the concatenated code $\mathscr{C}\mathscr{C}^{(2)}$ just introduced is different from the form $\mathscr{C}\mathscr{C}^{(1)}$ proposed in Section 3. For the concatenated code $\mathscr{C}\mathscr{C}^{(1)}$ the information vector $u_t^{(2)}$ of the inner code $\mathscr{C}_\mathrm{i}$ is the whole codeword $v_t^{(1)}$ of the outer code $\mathscr{C}_\mathrm{o}$; there, $u_t^{(2)} = v_t^{(1)}$ has m components. In contrast, for the concatenated code $\mathscr{C}\mathscr{C}^{(2)}$, the information vector $u_t^{(2)}$ of the inner code $\mathscr{C}_\mathrm{i}$ is only the parity vector $y_t^{(1)}$ of the codeword $v_t^{(1)}$ of the outer code $\mathscr{C}_\mathrm{o}$; in this case $u_t^{(2)} = y_t^{(1)}$ has $m-k$ components. So the parameters of the inner code of $\mathscr{C}\mathscr{C}^{(2)}$ are different from the parameters of the inner code of $\mathscr{C}\mathscr{C}^{(1)}$.

The next theorem introduces an input-state-output representation of the concatenated convolutional code $\mathscr{C}\mathscr{C}^{(2)}$ from an input-state-output representation of the outer and inner codes.

Theorem 9. *Let $\mathscr{C}_\mathrm{o}(A_1, B_1, C_1, D_1)$ be an (m, k, δ_1)-code and let $\mathscr{C}_\mathrm{i}(A_2, B_2, C_2, D_2)$ be an $(n-k, m-k, \delta_2)$-code. Then an input-state-output representation for the concatenated code $\mathscr{C}\mathscr{C}^{(2)}$ of rate k/n is given by system (1), where*

$$\begin{aligned} A &= \begin{pmatrix} A_2 & B_2C_1 \\ O & A_1 \end{pmatrix}, \quad B = \begin{pmatrix} B_2D_1 \\ B_1 \end{pmatrix}, \\ C &= \begin{pmatrix} C_2 & D_2C_1 \\ O & C_1 \end{pmatrix}, \quad D = \begin{pmatrix} D_2D_1 \\ D_1 \end{pmatrix}. \end{aligned} \tag{26}$$

Proof. By proceeding as in the proof of Theorem 2 we have

$$x_{t+1}^{(1)} = A_1 x_t^{(1)} + B_1 u_t^{(1)}, \quad x_{t+1}^{(2)} = A_2 x_t^{(2)} + B_2 u_t^{(2)},$$
$$y_t^{(1)} = C_1 x_t^{(1)} + D_1 u_t^{(1)}, \quad y_t^{(2)} = C_2 x_t^{(2)} + D_2 u_t^{(2)}.$$

Now, taking into account that $u_t^{(2)} = y_t^{(1)}$ and the comments at beginning of the section, an input-state-output representation of the concatenated code $\mathscr{CC}^{(2)}$ is given by

$$x_{t+1} = \begin{pmatrix} A_2 & B_2C_1 \\ O & A_1 \end{pmatrix} x_t + \begin{pmatrix} B_2D_1 \\ B_1 \end{pmatrix} u_t,$$
$$y_t = \begin{pmatrix} C_2 & D_2C_1 \\ O & C_1 \end{pmatrix} x_t + \begin{pmatrix} D_2D_1 \\ D_1 \end{pmatrix} u_t. \qquad \square$$

Using Theorems 1 and 9, the next theorem allows us to obtain the transfer function $T(z)$ associated to the concatenated code $\mathscr{CC}^{(2)}$ in terms of the transfer functions $T_1(z)$ and $T_2(z)$ associated to the outer code $\mathscr{C}_0$ and the inner code $\mathscr{C}_\mathrm{i}$, respectively.

Theorem 10. *Let $T_1(z)$ be the transfer function of the outer code $\mathscr{C}_\mathrm{o}$ and let $T_2(z)$ be the transfer function of the inner code $\mathscr{C}_\mathrm{i}$. Then the transfer function $T(z)$ associated to the concatenated code $\mathscr{CC}^{(2)}$ is*

$$T(z) = \begin{pmatrix} T_2(z)T_1(z) \\ T_1(z) \end{pmatrix}.$$

The next example shows that if pair (A_1, B_1) of the inner code and pair (A_2, B_2) of the outer code are controllable, then pair (A, B) of the concatenated code $\mathscr{CC}^{(2)}$ is not necessarily a controllable pair.

Example 10. As in Example 1, let α be a primitive element of the Galois field $\mathbb{F} = GF(8)$. Consider the $(2, 1, 1)$-outer code $\mathscr{C}_\mathrm{o}(A_1, B_1, C_1, D_1)$, where

$$A_1 = (0), \quad B_1 = (1), \quad C_1 = (\alpha^4), \quad D_1 = (\alpha^3)$$

and the $(2, 1, 1)$-inner code $\mathscr{C}_\mathrm{i}(A_2, B_2, C_2, D_2)$, where

$$A_2 = (\alpha), \quad B_2 = (1), \quad C_2 = (\alpha^4), \quad D_2 = (1)$$

(observe that this code is the outer code of Example 8). It follows then that the matrices (A_l, B_l) form a controllable pair, for $l = 1, 2$.

Now, from Theorem 9, the matrices A and B of an input-state-output representation of the concatenated code $\mathscr{CC}^{(2)}$ are

$$A = \begin{pmatrix} \alpha & \alpha^4 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha^3 \\ 1 \end{pmatrix}.$$

But (A, B) is not a controllable pair because $\operatorname{rank}(\alpha I - A \quad B) = 1 \neq 2$.

As in Theorem 4, the next theorem gives us conditions which ensure the controllability of pair (A, B) and the observability of pair (A, C). The proof is similar to the proof of Theorem 4 and it is omitted.

Theorem 11. *Let $\mathscr{C}_o(A_1, B_1, C_1, D_1)$ be an (m, k, δ_1)-code and let $\mathscr{C}_i(A_2, B_2, C_2, D_2)$ be an $(n-k, m-k, \delta_2)$-code. Let $\mathscr{C}\mathscr{C}^{(2)}(A, B, C, D)$ be the concatenated code described by the matrices in expression* (26).

(a) *If* $\mathrm{rank}(B) = \delta_1 + \delta_2$, *then (A, B, C, D) is a minimal representation of $\mathscr{C}\mathscr{C}^{(2)}$ having complexity $\delta_1 + \delta_2$.*
(b) *If pair (A_l, C_l) is observable for $l = 1, 2$, then pair (A, C) is observable.*

Now, as a consequence of Theorem 11 and Lemma 1, we have the following result.

Corollary 4. *Let $\mathscr{C}_o(A_1, B_1, C_1, D_1)$ be an (m, k, δ_1)-code and let $\mathscr{C}_i(A_2, B_2, C_2, D_2)$ be an $(n-k, m-k, \delta_2)$-code. Let $\mathscr{C}\mathscr{C}^{(2)}(A, B, C, D)$ be the concatenated code described by the matrices in expression* (26). *Assume that the following two conditions hold*:

(a) $\mathrm{Rank}(B) = \delta_1 + \delta_2$.
(b) *The matrices (A_l, C_l) form an observable pair, for $l = 1, 2$.*

Then (A, B, C, D) is a minimal representation of the observable convolutional code $\mathscr{C}\mathscr{C}^{(2)}$ with complexity $\delta_1 + \delta_2$.

Remark 8. Through a similar argument as in Remark 2, the condition (a) implies that pair (A_l, B_l) is controllable, for $l = 1, 2$.

Remark 9. Example 10 also shows that the converse of Remark 8 is not true because the pair (A_l, B_l) is controllable, for $l = 1, 2$, but $\mathrm{rank}(B) = 1 \neq 1 + 1$.

As in the previous model of concatenation, the next result provides the conditions for reaching the controllability of pair (A, B) of the concatenated code $\mathscr{C}\mathscr{C}^{(2)}$, for the particular case where the outer code has rate $1/2$ and complexity $\delta_1 = 1$ and the matrix A_2 is a diagonal matrix. The proof is similar to the proof in Theorem 5 replacing B_{21} with B_2, and we omit it.

Theorem 12. *Let $\mathscr{C}_o(A_1, B_1, C_1, D_1)$ be a $(2, 1, 1)$-code and let $\mathscr{C}_i(A_2, B_2, C_2, D_2)$ be an $(n-1, 1, \delta_2)$-code. Let $\mathscr{C}\mathscr{C}^{(2)}(A, B, C, D)$ be the concatenated code described by the matrices in expression* (26). *Assume the following conditions hold*:

(a) *A_2 is a diagonal matrix.*
(b) *The matrices (A_l, B_l) form a controllable pair.*
(c) *All the elements of the vector B_2 are nonzero.*
(d) $\mathrm{rank}\begin{pmatrix} -C_1 & D_1 \\ \lambda I - A_1 & B_1 \end{pmatrix} = 2$ *for all λ eigenvalue of A.*

Then, (A, B, C, D) is a minimal representation of the concatenated code $\mathscr{C}\mathscr{C}^{(2)}$ with complexity $\delta_1 + \delta_2 = 1 + \delta_2$.

Remark 10. Note that condition (d) of the previous theorem implies that the pair (A_1, C_1) is observable, since $\delta_1 = 1$ (see Remark 4).

For the particular case where the complexity of the outer code and the inner code is $\delta_1 = \delta_2 = 1$, we get a similar result to Theorem 6.

Theorem 13. *Let $\mathscr{C}_o(A_1, B_1, C_1, D_1)$ be an $(m, k, 1)$-code and let $\mathscr{C}_i(A_2, B_2, C_2, D_2)$ be an $(n-k, m-k, 1)$-code. Let $\mathscr{C}\mathscr{C}^{(2)}(A, B, C, D)$ be the concatenated code described by the matrices in expression* (26). *Assume the following conditions hold*:

(a) *The matrices (A_1, B_1) form a controllable pair.*
(b) *The matrix $\begin{pmatrix} -B_2C_1 & B_2D_1 \\ A_2 - A_1 & B_1 \end{pmatrix}$, of size $2 \times (k+1)$, has rank equal to* 2.

Then, (A, B, C, D) is a minimal representation of the concatenated code $\mathscr{C}\mathscr{C}^{(2)}(A, B, C, D)$ with complexity $\delta_1 + \delta_2 = 2$.

As a consequence of Theorem 13, we get the following result for the particular case where the matrices A_1 and A_2 are equal.

Corollary 5. *Let $\mathscr{C}_o(A_1, B_1, C_1, D_1)$ be an $(m, k, 1)$-code, and let $\mathscr{C}_i(A_2, B_2, C_2, D_2)$ be an $(n-k, m-k, 1)$-code. Let $\mathscr{C}\mathscr{C}^{(2)}(A, B, C, D)$ be the concatenated code described by the matrices in expression* (26). *Assume the following conditions hold*:

(a) $A_1 = A_2$.
(b) *The matrices (A_1, B_1) form a controllable pair.*
(c) *The vectors B_2 and C_1 are not orthogonal vectors.*

Then, (A, B, C, D) is a minimal representation for the concatenated code $\mathscr{C}\mathscr{C}^{(2)}$ with complexity $\delta_1 + \delta_2 = 2$.

Remark 11. Note that condition (c) of Corollary 5 implies in particular that the pair (A_2, B_2) is controllable and the pair (A_1, C_1) is observable, since taking into account that $\delta_1 = \delta_2 = 1$, the condition $B_2C_1 \neq O$ implies that $B_2 \neq O$ and $C_1 \neq O$.

If the outer code has rate $k/(k+1)$, then the matrices B_2 and C_1 are of size 1×1. Then, the vectors B_2 and C_1 are orthogonal if and only if one of them is the zero vector; so it is sufficient to have the controllability of the pair (A_2, B_2) and the observability of (A_1, C_1) so that condition (c) of Corollary 5 holds. Then, we get the following result.

Corollary 6. *Let $\mathscr{C}_o(A_1, B_1, C_1, D_1)$ be a $(k+1, k, 1)$-code and let $\mathscr{C}_i(A_2, B_2, C_2, D_2)$ be an $(n-k, 1, 1)$-code. Let $\mathscr{C}\mathscr{C}^{(2)}(A, B, C, D)$ be the concatenated code described by the matrices in expression* (26). *Assume the following conditions hold*:

(a) $A_1 = A_2$.
(b) *The matrices (A_l, B_l) form a controllable pair, for $l = 1, 2$.*
(c) *The matrices (A_1, C_1) form an observable pair.*

Then, (A, B, C, D) is a minimal representation of the concatenated code $\mathscr{C}\mathscr{C}^{(2)}$ with complexity $\delta_1 + \delta_2 = 2$.

The proof of the following theorem is similar to the proof of Theorem 7 and we omit it.

Theorem 14. *Let $\mathscr{C}_o(A_1, B_1, C_1, D_1)$ be a $(2, 1, \delta_1)$-code and let $\mathscr{C}_i(A_2, B_2, C_2, D_2)$ be an $(n-1, 1, \delta_2)$-code. Let $\mathscr{CC}^{(2)}(A, B, C, D)$ be the concatenated code described by the matrices in expression (26). Assume the following conditions hold*:

(a) $\sigma(A_2) \subseteq \sigma(A_1)$.
(b) *The matrices (A_l, B_l) form a controllable pair, for $l = 1, 2$.*
(c) *All the elements of matrix B_2 are nonzero.*
(d) *The matrices (A_l, C_l) form an observable pair, for $l = 1, 2$.*

Then (A, B, C, D) is a minimal representation of the concatenated code $\mathscr{CC}^{(2)}$ with complexity $\delta_1 + \delta_2$. Furthermore, $\mathscr{CC}^{(2)}$ is an observable convolutional code.

If in Theorem 14 we do not require the controllability condition of one of the pairs (A_l, B_l), for $l = 1, 2$, then the representation of the concatenated code $\mathscr{CC}^{(2)}$ may be nonminimal, as we can see in the following example.

Example 11. Consider the outer code $\mathscr{C}_o(A_1, B_1, C_1, D_1)$ described by matrices A_1, B_1 and C_1 of Example 1 and

$$D_1 = \begin{pmatrix} 1 & \alpha^5 \end{pmatrix}$$

and consider the $(2, 1, 1)$-code $\mathscr{C}_i(A_2, B_2, C_2, D_2)$, where

$$A_2 = \begin{pmatrix} \alpha^4 & 1 \\ \alpha^3 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ \alpha \end{pmatrix},$$

where C_2 and D_2 are arbitrary matrices, so that the matrices (A_2, C_2) form an observable pair.

Observe that A_2 and B_2 are the matrices A_2 and B_{21}, respectively, of Example 5. So, (A_2, B_2) is not a controllable pair. Furthermore, $\sigma(A_2) = \{\alpha, \alpha^2\} = \sigma(A_1)$, so condition (a) of Theorem 14 is verified.

Since the matrices A and B of an input-state-output representation of the concatenated code $\mathscr{CC}^{(2)}$ are

$$A = \begin{pmatrix} \alpha^4 & 1 & \alpha^4 & \alpha^3 \\ \alpha^3 & 0 & \alpha^5 & \alpha^4 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \alpha^5 \\ \alpha & \alpha^6 \\ 1 & 0 \\ 0 & \alpha^6 \end{pmatrix},$$

we have that (A, B) is not a controllable pair because

$$\operatorname{rank}(\alpha I - A \quad B) = \operatorname{rank} \begin{pmatrix} \alpha^2 & 1 & \alpha^4 & \alpha^3 & 1 & \alpha^5 \\ \alpha^3 & \alpha & \alpha^5 & \alpha^4 & \alpha & \alpha^6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha^4 & 0 & \alpha^6 \end{pmatrix} = 3 \neq 4.$$

Now, as in the concatenated code $\mathscr{CC}^{(1)}$, Theorem 14 does not verify for the case where $\sigma(A_2) \not\subseteq \sigma(A_1)$, as we can see in the following example.

Example 12. As in Example 1, let α be a primitive element of $\mathbb{F} = GF(8)$. Let $\mathscr{C}_o(A_1, B_1, C_1, D_1)$ be the $(2, 1, 1)$-code described by matrices

$$A_1 = (\alpha^2), \quad B_1 = (\alpha), \quad C_1 = (1), \quad D_1 = (\alpha)$$

and let $\mathscr{C}_\mathrm{i}(A_2, B_2, C_2, D_2)$ be an (2, 1, 2)-code, where

$$A_2 = \begin{pmatrix} \alpha^2 & 0 \\ 1 & \alpha^6 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \alpha \\ 1 \end{pmatrix},$$

and C_2 and D_2 are arbitrary matrices of sizes 1×2 and 1×1, respectively, so that (A_2, C_2) is an observable pair. Observe that $\mathscr{C}_\mathrm{o}(A_1, B_1, C_1, D_1)$ is the outer code of Example 6 and the matrices A_2 and B_2 are the matrices A_2 and B_{21} respectively of Example 6. So (A_l, B_l) is a controllable pair, for $l = 1, 2$, (A_1, C_1) is an observable pair and $\sigma(A_2) \nsubseteq \sigma(A_1)$.

Now, taking into account Theorem 9, the matrices A and B of an input-state-output representation of $\mathscr{C}\mathscr{C}^{(2)}$, are given by

$$A = \begin{pmatrix} \alpha^2 & 0 & \alpha \\ 1 & \alpha^6 & 1 \\ 0 & 0 & \alpha^2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha^2 \\ \alpha \\ \alpha \end{pmatrix}.$$

These matrices are the matrices A and B of Example 6, and then (A, B) is not a controllable pair.

The next example shows that we cannot obtain a lower bound in terms of $d_{\mathrm{free}}(\mathscr{C}_\mathrm{i})$ as in expression (20) for the concatenated code $\mathscr{C}\mathscr{C}^{(2)}$.

Example 13. As in Example 1, let α be a primitive element of the field $\mathbb{F} = GF(8)$. Consider the (3, 2, 1)-outer code $\mathscr{C}_\mathrm{o}(A_1, B_1, C_1, D_1)$ and the (2, 1, 1)-inner code $\mathscr{C}_\mathrm{i}(A_2, B_2, C_2, D_2)$ where

$$A_1 = (\alpha^2), \quad B_1 = (1 \quad 1), \quad C_1 = (1), \quad D_1 = (1 \quad 1),$$
$$A_2 = (\alpha), \quad B_2 = (1), \quad C_2 = (\alpha^4), \quad D_2 = (1).$$

Observe that the outer code is the inner code of Example 8 and the inner code is the outer code of Example 8. Consequently, for $l = 1, 2$, the pairs (A_l, B_l) and (A_l, C_l) are controllable and observable, respectively.

So, by applying Theorem 9, Theorem 13 and part (b) of Theorem 11, the matrices

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha^4 & 1 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

are a minimal representation of the (4, 2, 2)-observable code $\mathscr{C}\mathscr{C}^{(2)}$.

In addition $d_{\mathrm{free}}(\mathscr{C}_\mathrm{o}) = 2$ and $d_{\mathrm{free}}(\mathscr{C}_\mathrm{i}) = 4$. Therefore

$$\max\{d_{\mathrm{free}}(\mathscr{C}_\mathrm{o}), d_{\mathrm{free}}(\mathscr{C}_\mathrm{i})\} = 4.$$

Nevertheless,

$$d_{\mathrm{free}}(\mathscr{C}\mathscr{C}^{(2)}) = 2.$$

Observe that, in this case, $\operatorname{rank}(D_1) = \operatorname{rank}(1 \quad 1) = 1 = m - k$ and $k = 2$.

As a consequence of the previous example, we can only obtain a lower bound for the free distance of $\mathscr{C}\mathscr{C}^{(2)}$ in terms of $d_{\mathrm{free}}(\mathscr{C}_\mathrm{o})$. Nevertheless, if we require $\operatorname{rank}(D_1) = k$, then we can obtain a refinement of this bound as we prove in the following theorem. Firstly, we obtain a lower bound for the column distances of $\mathscr{C}\mathscr{C}^{(2)}$ in terms of the column distances of $\mathscr{C}_\mathrm{o}$ and $\mathscr{C}_\mathrm{i}$.

Lemma 3. *Let $\mathscr{C}\mathscr{C}^{(2)}$ be the concatenated code given by Theorem 9 from the outer code $\mathscr{C}_{\mathrm{o}}$ and the inner code $\mathscr{C}_{\mathrm{i}}$. Then*

(a) $d_j^c(\mathscr{C}\mathscr{C}^{(2)}) \geqslant d_j^c(\mathscr{C}_{\mathrm{o}})$ *for* $j = 0, 1, 2, \ldots$
(b) *If* $\mathrm{rank}(D_1) = k$, *then* $d_j^c(\mathscr{C}\mathscr{C}^{(2)}) \geqslant d_j^c(\mathscr{C}_{\mathrm{i}}) + 1$.

Proof. Taking into account the relations between $y_t, y_t^{(1)}, y_t^{(2)}$; $u_t, u_t^{(1)}, u_t^{(2)}$, and $v_t, v_t^{(1)}, v_t^{(2)}$ given by expressions (21)–(25) and using the same argument used to obtain inequality (18) in the proof of Lemma 2, we obtain inequality of part (a).

Now, since $y_0^{(1)} = D_1 u_0^{(1)}$ and $\mathrm{rank}(D_1) = k$, we get that $u_0 = u_0^{(1)} \neq 0$ if and only if $u_0^{(2)} = y_0^{(1)} \neq 0$. So, from expressions (5) and (21)–(25), we have

$$\begin{aligned} d_j^c(\mathscr{C}\mathscr{C}^{(2)}) = \min_{u_0 \neq 0} \left\{ \sum_{t=0}^{j} \mathrm{wt}(v_t) \right\} &= \min_{u_0^{(1)} \neq 0} \left\{ \sum_{t=0}^{j} \mathrm{wt}\big(u_t^{(1)}\big) \right\} \\ &+ \min_{u_0^{(2)} \neq 0} \left\{ \sum_{t=0}^{j} \mathrm{wt}\big(v_t^{(2)}\big) \right\} \geqslant 1 + d_j^c(\mathscr{C}_{\mathrm{i}}). \qquad \square \end{aligned}$$

Now, as an immediate consequence of expression (4) and the above lemma we obtain the following result.

Theorem 15. *Let $\mathscr{C}\mathscr{C}^{(2)}$ be the concatenated code given by Theorem 9 from the outer code $\mathscr{C}_{\mathrm{o}}$ and the inner code $\mathscr{C}_{\mathrm{i}}$. Then*

(a) $d_{\mathrm{free}}(\mathscr{C}\mathscr{C}^{(2)}) \geqslant d_{\mathrm{free}}(\mathscr{C}_{\mathrm{o}})$.
(b) *If* $\mathrm{rank}(D_1) = k$, *then* $d_{\mathrm{free}}(\mathscr{C}\mathscr{C}^{(2)}) \geqslant d_{\mathrm{free}}(\mathscr{C}_{\mathrm{i}}) + 1$.

As in the previous section, we finish this section with some examples. In both examples we use the construction proposed by Smarandache and Rosenthal [28] to obtain an MDS convolutional code.

Example 14. As in Example 1, let α be a primitive element of $\mathbb{F} = GF(8)$. Let $\mathscr{C}_{\mathrm{o}}(A_1, B_1, C_1, D_1)$ and $\mathscr{C}_{\mathrm{i}}(A_2, B_2, C_2, D_2)$ be the $(2, 1, 1)$-codes where

$$A_l = (\alpha), \quad B_l = (1), \quad C_l = (\alpha^4), \quad D_l = (1)$$

for $l = 1, 2$. Since $\mathscr{C}_{\mathrm{o}}$ and therefore $\mathscr{C}_{\mathrm{i}}$ are MDS convolutional codes, $d_{\mathrm{free}}(\mathscr{C}_{\mathrm{o}}) = d_{\mathrm{free}}(\mathscr{C}_{\mathrm{i}}) = 4$ (see Example 8).

It follows then that the matrices (A_l, B_l) form a controllable pair, for $l = 1, 2$ and the matrices (A_l, C_l) form an observable pair, for $l = 1, 2$. So, from Theorem 9, Corollary 6 and part (b) of Theorem 11, the matrices

$$A = \begin{pmatrix} \alpha & \alpha^4 \\ 0 & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha^4 & \alpha^4 \\ 0 & \alpha^4 \end{pmatrix}, \quad D = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

are a minimal representation of the observable code $\mathscr{C}\mathscr{C}^{(2)}$.

Now, since $\operatorname{rank}(D_1) = 1$, from Theorem 15 and the Singleton bound, we have

$$5 = \max\{4, 4 + 1\} \leqslant d_{\text{free}}(\mathscr{C}\mathscr{C}^{(2)}) \leqslant 9.$$

But in fact, $d_{\text{free}}(\mathscr{C}\mathscr{C}^{(2)}) = 7$.

Example 15. As in Example 1, let α be a primitive element of $\mathbb{F} = GF(8)$. Let $\mathscr{C}_{\text{o}}(A_1, B_1, C_1, D_1)$ be the MDS convolutional code of rate $1/2$ and degree $\delta_1 = 2$, where

$$A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, \quad C_1 = (\alpha^3 \quad \alpha^4), \quad D_1 = (1).$$

In this case, $\mathscr{C}_{\text{o}}$ is an MDS convolutional code, so $d_{\text{free}}(\mathscr{C}_{\text{o}}) = 6$.

Let $\mathscr{C}_{\text{i}}(A_2, B_2, C_2, D_2)$ be the MDS convolutional code of rate $1/2$ and degree $\delta_2 = 1$, where

$$A_2 = (\alpha), \quad B_2 = (1), \quad C_2 = (\alpha^2), \quad D_2 = (\alpha^3).$$

It follows then that the matrices (A_l, B_l) form a controllable pair, for $l = 1, 2$ and the matrices (A_l, C_l) form an observable pair, for $l = 1, 2$. Furthermore, $\sigma(A_2) \subseteq \sigma(A_1)$. So, by applying Theorem 9, Theorem 14 and part (b) of Theorem 11, the matrices

$$A = \begin{pmatrix} \alpha & \alpha^3 & \alpha^4 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ \alpha \end{pmatrix}, \quad C = \begin{pmatrix} \alpha^2 & \alpha^6 & 1 \\ 0 & \alpha^3 & \alpha^4 \end{pmatrix}, \quad D = \begin{pmatrix} \alpha^3 \\ 1 \end{pmatrix}$$

are a minimal representation of the observable code $\mathscr{C}\mathscr{C}^{(2)}$.

Now, since $\operatorname{rank}(D_1) = 1$, by Theorem 15 and the Singleton bound, we have

$$6 = \max\{6, 4 + 1\} \leqslant d_{\text{free}}(\mathscr{C}\mathscr{C}^{(2)}) \leqslant 12.$$

But in fact, $d_{\text{free}}(\mathscr{C}\mathscr{C}^{(2)}) = 11$.

Now, if we permute the outer and the inner code, then applying Theorem 9 and Theorem 12 we obtain a minimal representation of the new concatenated code $\mathscr{C}\mathscr{C}^{(2)}$ given by

$$A = \begin{pmatrix} \alpha & 0 & \alpha^2 \\ 0 & \alpha^2 & \alpha^3 \\ 0 & 0 & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} \alpha^3 \\ \alpha^4 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha^3 & \alpha^4 & \alpha^2 \\ 0 & 0 & \alpha^2 \end{pmatrix}, \quad D = \begin{pmatrix} \alpha^3 \\ \alpha^3 \end{pmatrix}$$

which is observable by using part (b) of Theorem 11.

Now, since $\operatorname{rank}(D_1) = 1$, through Theorem 15 and the Singleton bound, we have

$$7 = \max\{4, 6 + 1\} \leqslant d_{\text{free}}(\mathscr{C}\mathscr{C}^{(2)}) \leqslant 12.$$

But in fact, $d_{\text{free}}(\mathscr{C}\mathscr{C}^{(2)}) = 12$. So we get an MDS convolutional code.

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